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SPATIAL STATIONARY LONG WAVES IN SHEAR FLOWS

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The system of integrodifferential equations describing the spatial stationary free-boundary shear flows of an ideal fluid in the shallow-water approximation is considered. The generalized characteristics of the model are found and the hyperbolicity conditions are formulated. A new class of exact solutions of the governing equations is obtained which is characterized by a special dependence of the desired functions on the vertical coordinate. The system of equations describing this class of solutions in the hyperbolic case is reduced to Riemann invariants. New exact solutions of the equations of motion are found.

Key words: *nonlinear waves, shallow water, rotational flow.*

Classical shallow-water theory has been widely used in the modeling of wave processes. A mathematical justification of this approximate model was developed by Ovsyannikov [1] (see also [2]). The model taking into account the shear nature of the motion has been studied to a lesser degree. The approach used in the present study is based on the extension of the theory of characteristics for systems of integrodifferential equations as proposed in [3, 4]. This allows one to analyze the possible types of waves on shear flows busing the analogy with the classical case.

1. Formulation of the Problem. We consider the long-wave approximation of the stationary Euler equations

$$(\boldsymbol{U}\cdot\nabla)u + p_x/\rho = 0, \qquad (\boldsymbol{U}\cdot\nabla)v + p_y/\rho = 0,$$
$$p_z = -\rho g, \quad \operatorname{div}\boldsymbol{U} = 0, \tag{1.1}$$

which is derived from the exact equations governing the motion of an ideal incompressible heavy fluid by using an asymptotic expansion in the small parameter $\varepsilon = H_0/L_0$ (H_0 is the characteristic vertical scale and L_0 is the characteristic horizontal scale; $H_0/L_0 \ll 1$). In (1.1), $\boldsymbol{U} = (u, v, w)$ is the fluid velocity, p is the pressure, $\rho = \text{const}$ is the fluid density, and x, y, and z are Cartesian coordinates in space.

The approximate expression for the vorticity vector $\boldsymbol{\Omega}$ is obtained from the standard expression $(w_y - v_z, u_z - w_x, v_x - u_y)$ by neglecting small terms of order ε or higher:

$$\boldsymbol{\Omega} = (-v_z, u_z, v_x - u_y).$$

By virtue of (1.1), the vector $\boldsymbol{\Omega}$ satisfies the equation

$$\boldsymbol{\Omega}_t + (\boldsymbol{U}\cdot\nabla)\boldsymbol{\Omega} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{U}. \tag{1.2}$$

From (1.2) it follows that if $u_z = 0$ and $v_z = 0$ at the initial time, then $u_z = 0$ and $v_z = 0$ at all times. The class of solutions characterized by the inequality $u_z^2 + v_z^2 \neq 0$ will be called *shear flows*. In the following, we consider free-boundary shear flows in a layer $0 \leqslant z \leqslant h(x,y)$. On the free surface, $z = h(x,y)$, the pressure is constant, $p = p_0 = \text{const}$. In addition, the kinematic condition

$$\Big(\int\limits_0^h u\,dz\Big)_x + \Big(\int\limits_0^h v\,dz\Big)_y = 0$$

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and the nonpenetration condition $w = 0$ on the even bottom $z = 0$ should be satisfied. Taking into account the boundary conditions, we find the pressure distribution in the fluid (the hydrostatic law):

$$p = p_0 + \rho g(h - z). \tag{1.3}$$

Substitution of (1.3) into (1.1) yields the equations

$$(\boldsymbol{U} \cdot \nabla)u + gh_x = 0, \qquad (\boldsymbol{U} \cdot \nabla)u + gh_y = 0,$$

$$w = -\int_0^z (u_x + v_y)\, dz', \qquad \Big(\int_0^h u\, dz\Big)_x + \Big(\int_0^h v\, dz\Big)_y = 0, \tag{1.4}$$

which represent an extension of the classical shallow-water equations to shear flows. The classical model

$$uu_x + vu_y + gh_x = 0, \qquad uv_x + vv_y + gh_y = 0,$$

$$w = -z(u_x + v_y), \qquad (hu)_x + (hv)_y = 0$$

describes particular solutions of Eqs. (1.4) — stationary shearless flows, for which $u_z = 0$ and $v_z = 0$. We note that stationary plane-parallel shear flow were studied in [5].

To formulate the initial-boundary-value problem for a fixed domain, we convert to mixed Eulerian–Lagrangian coordinates x', y', and λ:

$$x' = x, \qquad y' = y, \qquad \Phi(x', y', \lambda) = z.$$

Here the function $\Phi(x', y', \lambda)$ is a solution of the Cauchy problem

$$u(x, y, \Phi)\Phi_x + v(x, y, \Phi)\Phi_y = w(x, y, \Phi), \qquad \Phi\Big|_{x=0} = \Phi_0(y, \lambda);$$

it is assumed that $u(0, y, \Phi_0(y, \lambda)) \neq 0$, where $\lambda \in [0, 1]$. The function $\Phi_0(y, \lambda)$ is chosen such that $\lambda = 0$ corresponds to the even bottom $[\Phi_0(y, 0) = 0]$ and $\lambda = 1$ to the free surface $[\Phi_0(y, 1) = h(0, y)]$. Then, the equalities $\Phi_0(x, y, 0) = 0$ and $\Phi_0(x, y, 1) = h(x, y)$ are valid for all x. Therefore, in the new variables, the region occupied by the fluid is the fixed layer $0 \leqslant \lambda \leqslant 1$.

Equations (1.4) become

$$(\boldsymbol{u} \cdot \nabla)u + g\Big(\int_0^1 H\, d\nu\Big)_x = 0, \qquad (\boldsymbol{u} \cdot \nabla)v + g\Big(\int_0^1 H\, d\nu\Big)_y = 0,$$
$$(uH)_x + (vH)_y = 0, \tag{1.5}$$

where $H = \Phi_\lambda(x, y, \lambda)$ and $\boldsymbol{u} = (u(x, y, \lambda), v(x, y, \lambda), 0)$. Once system (1.5) is solved, one can find

$$\Phi = \int_0^\lambda H\, d\nu, \qquad w = u\Phi_x + v\Phi_y.$$

Equations (1.5) can be written in operator-differential form

$$A\boldsymbol{V}_x + B\boldsymbol{V}_y = 0, \qquad \boldsymbol{V} = (u, v, H)^{\mathrm{t}}, \tag{1.6}$$

where the operators A and B are given by

$$A = \begin{pmatrix} u, & 0, & g\displaystyle\int_0^1 \ldots\, d\nu \\ 0, & u, & 0 \\ H, & 0, & u \end{pmatrix}, \qquad B = \begin{pmatrix} v, & 0, & 0 \\ 0, & v, & g\displaystyle\int_0^1 \ldots\, d\nu \\ 0, & H, & v \end{pmatrix}.$$

Here the action of the operator $J = \int\limits_0^1 \ldots d\nu$ on the function $f(x, y, \lambda)$ is defined by

$$(Jf)(x, y) = \int\limits_0^1 f(x, y, \nu)\, d\nu.$$

Let us find conditions under which system (1.6) is a generalized hyperbolic one. Below, we use the following definitions (see [3, 6]).

1. A curve Γ in the plane of the variables x and y with the normal $\boldsymbol{n} = (\xi, \eta)$ is a characteristic of system (1.6) if the problem

$$(\boldsymbol{F}, (\xi A + \eta B)\boldsymbol{\varphi}) = 0 \tag{1.7}$$

has a nontrivial solution $\boldsymbol{F}$. Here $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^{\rm t}$ is an arbitrary vector function which is smooth in the variable λ and $\boldsymbol{F} = (F_1, F_2, F_3)$ is the desired eigenfunctional (acting on functions of the variable λ); $(\boldsymbol{F}, \boldsymbol{\varphi})$ denotes the result of action of the functional $\boldsymbol{F}$ on the trial function $\boldsymbol{\varphi}$.

2. The equality

$$(\boldsymbol{F}, A\boldsymbol{V}_x + B\boldsymbol{V}_y) = 0$$

is called *the characteristic relation* (*condition*).

3. System (1.6) is called *hyperbolic in the direction of the vector* $\boldsymbol{\mu} = (\mu_1, \mu_2)$, $|\boldsymbol{\mu}| = 1$ if for any vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ orthogonal to $\boldsymbol{\mu}$, the problem

$$(\boldsymbol{F}, (\zeta(\mu_1 A + \mu_2 B) + \sigma_1 A + \sigma_2 B)\boldsymbol{\varphi}) = 0$$

($\boldsymbol{n} = \zeta\boldsymbol{\mu} + \boldsymbol{\sigma}$) has nontrivial solutions $\boldsymbol{F} = \boldsymbol{F}^\alpha$ only for real $\zeta = \zeta^\alpha$ and the set of eigenfunctionals $\{\boldsymbol{F}^\alpha\}$ has the completeness property [if the equality $\boldsymbol{F}^\alpha$ holds for any $(\boldsymbol{F}^\alpha, \boldsymbol{\varphi}) = 0$, then $\boldsymbol{\varphi} = 0$]. We note that here $\{\boldsymbol{F}^\alpha\}$ can be both regular functionals represented by locally integrable functions and singular functionals belonging to the space of generalized functions.

2. Characteristics of the System of Stationary Long Waves. Relation (1.7) leads to the following equation for the vector functional $\boldsymbol{F} = (F_1, F_2, F_3)$:

$$(\boldsymbol{F}, (\xi A + \eta B)\boldsymbol{\varphi}) = (F_1, (\xi u + \eta v)\varphi_1) + \xi g \int\limits_0^1 \varphi_3\, d\nu (F_1, 1) + (F_2, (\xi u + \eta v)\varphi_2)$$
$$+ \eta g \int\limits_0^1 \varphi_3\, d\nu (F_2, 1) + \xi(F_3, H\varphi_1) + \eta(F_3, H\varphi_2) + (F_3, (\xi u + \eta v)\varphi_3) = 0.$$

By virtue of the independence of the trial functions φ_i, the following equalities hold:

$$(F_1, (\xi u + \eta v)\varphi_1) + \xi(F_3, H\varphi_1) = 0, \qquad (F_2, (\xi u + \eta v)\varphi_2) + \eta(F_3, H\varphi_2) = 0,$$
$$(F_3, (\xi u + \eta v)\varphi_3) + g\int\limits_0^1 \varphi_3\, d\nu (\eta F_2 + \xi F_1, 1) = 0. \tag{2.1}$$

Combining the first two relations, we obtain

$$(\eta F_1 - \xi F_2, (\xi u + \eta v)\varphi) = 0; \tag{2.2}$$

$$(\eta F_2 + \xi F_1, (\xi u + \eta v)\varphi) + (\xi^2 + \eta^2)(F_3, H\varphi) = 0. \tag{2.3}$$

Because $H \neq 0$, the action of F_3 on an arbitrary smooth function φ is defined by the formula

$$(F_3, \varphi) = -\frac{1}{\xi^2 + \eta^2}\Big(\eta F_2 + \xi F_1, \frac{\xi u + \eta v}{H}\,\varphi\Big), \tag{2.4}$$

which follows from (2.3). Next, using the third equation of system (2.1), we obtain the problem for the unknown functional $\eta F_2 + \xi F_1$:

$$\Big(\eta F_2 + \xi F_1, \frac{(\xi u + \eta v)^2}{H}\,\varphi\Big) - g(\xi^2 + \eta^2)\int\limits_0^1 \varphi\, d\nu (\eta F_2 + \xi F_1, 1) = 0. \tag{2.5}$$

Let us find the solutions of problem (2.1) in the class of generalized functions. For a fixed value $\lambda \in (0,1)$, we choose a vector $\boldsymbol{n} = (\xi, \eta)$ orthogonal to the vector $(u(\lambda), v(\lambda))$ [ξ and η satisfies the relation $-\xi/\eta = \tan\,\theta(\lambda)$, θ and q define polar coordinates in the plane (u, v), $u = q\cos\theta$, and $v = q\sin\theta$]. In the following, it is assumed that θ is a monotonic function of λ; for definiteness, $\theta_\lambda > 0$. Here and below, the arguments x and y of the functions u, v, and θ are omitted.

Obviously, Eqs. (2.2)–(2.5) are satisfied if we set

$$\eta F_1 - \xi F_2 = \delta(\nu - \lambda), \qquad \eta F_2 + \xi F_1 = 0, \qquad F_3 = 0. \tag{2.6}$$

Let us find the vector $\boldsymbol{\mu}$ present in the definition of the direction of hyperbolicity. According to this definition, it is necessary to find a vector $\boldsymbol{\mu}$ such that for any λ, the equation $\zeta(\mu_1 u(\lambda) + \mu_2 v(\lambda)) + \sigma_1 u(\lambda) + \sigma_2 v(\lambda) = 0$ is uniquely solvable for ζ. Obviously, this condition implies that the inequality $\mu_1 u(\lambda) + \mu_2 v(\lambda) \neq 0$ should be valid for all values of λ. Let $\mu_1 = -\sin\gamma$ and $\mu_2 = \cos\gamma$. The previous inequality becomes $\sin(\theta - \gamma) \neq 0$. Such γ exists only if $\theta_{\max} \geqslant \theta(\lambda) \geqslant \theta_{\min}$ and

$$\theta_{\max} - \theta_{\min} < \pi. \tag{2.7}$$

Indeed, if this inequality is valid, then, obviously, $\sin(\theta - \gamma) \neq 0$ for γ belonging to the interval $(\theta_1, \theta_0 + \pi)$ and to all intervals obtained by shifting this interval by $k\pi$, where k is an integer. If $\theta_{\max} - \theta_{\min} > \pi$, then for any γ, there exists $\theta \in (\theta_{\min}, \theta_{\max})$ such that $\sin(\theta - \gamma) = 0$. The above reasoning shows that inequality (2.7) is a necessary condition for the hyperbolicity of Eqs. (1.6).

Subsequently, in constructing the eigenfunctionals at a fixed point (x_0, y_0) on plane (x, y), we shall choose a Cartesian basis with origin at (x_0, y_0) so that $\theta_{\max} = -\theta_{\min}$. In a neighborhood of the point (x_0, y_0), the characteristic equations are specified by $y = y(x)$ and denoted by $k = dy/dx$. Setting $\xi = -k$ and $\eta = 1$ in the previous formulas, from (2.6) we obtain the components

$$F_1^{1\lambda} = \delta(\nu - \lambda), \qquad F_2^{1\lambda} = k^\lambda \delta(\nu - \lambda), \qquad F_3^{1\lambda} = 0$$

of the eigenfunctional $\boldsymbol{F}^{1\lambda} = (F_1^{1\lambda}, F_2^{1\lambda}, F_3^{1\lambda})$ corresponding to the eigenvalue $k^\lambda(x, y) = \tan\,\theta(x, y, \lambda)$ (λ is fixed). The second vector functional $\boldsymbol{F}^{2\lambda} = (F_1^{2\lambda}, F_2^{2\lambda}, F_3^{2\lambda})$ corresponding to the value $k^\lambda(x, y) = \tan\,\theta(x, y, \lambda)$, is determined using (2.4) and the relations

$$F_2 - \tan\,\theta(\lambda) F_1 = (1 + \tan^2\theta(\lambda))\delta'(\nu - \lambda), \qquad F_1 + \tan\,\theta(\lambda) F_2 = 0.$$

Direct substitution shows that

$$F_1^{2\lambda} = -\tan\,\theta(\lambda)\delta'(\nu - \lambda), \qquad F_2^{2\lambda} = \delta'(\nu - \lambda), \qquad F_3^{2\lambda} = \frac{v_\lambda - \tan\,\theta(\lambda) u_\lambda}{H}\,\delta(\nu - \lambda)$$

satisfy Eqs. (2.1). In these formulas, $\delta(\nu - \lambda)$ is the Dirac delta function acting on a smooth function $\varphi(\nu)$ by a the rule $(\delta(\nu - \lambda), \varphi(\nu)) = \varphi(\lambda)$, and $\delta'(\nu - \lambda)$ is its derivative acting on smooth functions by the rule $(\delta'(\nu - \lambda), \varphi(\nu)) = -\varphi_\lambda(\lambda)$.

To construct one more solution of the eigenvalue problem, we introduce the functional P^λ acting on a smooth function φ by the rule

$$(P^\lambda, \varphi(\nu)) = \int_0^1 \frac{H'(\varphi(\nu) - \varphi(\lambda))}{(v' - u'\tan\,\theta)^2}\,d\nu.$$

Here and below, the quantities with a prime depend on ν and those without a prime depend on the variable λ. The integral in the above formula is calculated in the sense of the principal value. Equation (2.5) holds for $\boldsymbol{n} = (\xi, \eta) = (-\tan\,\theta(\lambda), 1)$ if

$$F_2 - \tan\,\theta(\lambda) F_1 = \delta(\nu - \lambda) + g(1 + \tan^2\theta(\lambda))P^\lambda. \tag{2.8}$$

The components of the third vector functional $\boldsymbol{F}^{3\lambda}$ corresponding to the same eigenvalue $k^\lambda = \tan\,\theta(\lambda)$ are determined using relations (2.4) and (2.8) and the additional equality

$$F_1 + \tan\,\theta(\lambda) F_2 = 0.$$

As a result, we obtain

$$F_1^{3\lambda} = -\frac{\tan\,\theta(\lambda)}{1+\tan^2\theta(\lambda)}\,\delta(\nu-\lambda) - g\tan\,\theta(\lambda)P^\lambda, \qquad F_2^{3\lambda} = \frac{1}{1+\tan^2\theta(\lambda)}\,\delta(\nu-\lambda) + gP^\lambda,$$

$$(F_3^{3\lambda},\varphi) = -g\int\limits_0^1 \frac{\varphi'\,d\nu}{v'-u'\tan\,\theta}.$$

Let us determine the eigenvalues k outside the range of the function $\tan\,\theta(\lambda)$ ($v-ku\neq 0$ for any $\lambda\in[0,1]$) and the corresponding eigenfunctionals. Equation (2.2) yields the equality $F_1+kF_2=0$. Equation (2.5) implies that

$$(F_2-kF_1,\varphi) = g(1+k^2)\int\limits_0^1 \frac{\varphi H'd\nu}{(v'-ku')^2}\,(F_2-kF_1,1).$$

Setting $\varphi=1$ in this equality, we obtain the characteristic equation for k:

$$1 = g(1+k^2)\int\limits_0^1 \frac{H'\,d\nu}{(v'-ku')^2}. \tag{2.9}$$

Taking into account that F_2-kF_1 is determined with accuracy up to a factor, we choose this factor such that $(F_2-kF_1,1)=1$. Then, using (2.4) and (2.6), we find the components of the vector functionals $\boldsymbol{F}^i=(F_1^i,F_2^i,F_3^i)$ corresponding to the roots k^i of the characteristic equation (2.9):

$$(F_1^i,\varphi) = -k^ig\int\limits_0^1 \frac{H'\varphi'}{(v'-k^iu')^2}\,d\nu, \qquad (F_2^i,\varphi) = g\int\limits_0^1 \frac{H'\varphi'}{(v'-k^iu')^2}\,d\nu, \qquad (F_3^i,\varphi) = -g\int\limits_0^1 \frac{\varphi'}{v'-k^iu'}\,d\nu.$$

3. Characteristic Equation. The characteristic equation (2.9) can be written as

$$\chi(k) = g\int\limits_0^1 \frac{(1+k^2)H'\,d\nu}{u'^2(\tan\,\theta'-k)^2} - 1 = \tilde\chi(\gamma) = g\int\limits_0^1 \frac{H'\,d\theta'}{\theta'_\nu q'^2\sin^2(\theta'-\gamma)} - 1 = 0.$$

Here the quantity k and the variable γ are linked by the relation $k=\tan\,\gamma$. We note that $\tilde\chi(\gamma)=\tilde\chi(\gamma\pm\pi)$ and $\tilde\chi(\gamma)\to\infty$ for $\gamma\to\theta_1$ and for $\gamma\to\theta_0+\pi$. Since

$$\tilde\chi''(\gamma) = g\int\limits_0^1 \frac{(\sin^2(\theta'-\gamma)+3\cos^2(\theta'-\gamma))H'\,d\theta'}{\theta'_\nu q'^2\sin^4(\theta'-\gamma)} > 0,$$

it can be shown that on the interval $(\theta_1,\theta_0+\pi)$ the function $\tilde\chi(\gamma)$ has the single minimum at the point $\gamma_*\in(\theta_1,\theta_0+\pi)$ where $\tilde\chi'(\gamma_*)=0$. Note that if $\tilde\chi(\gamma_*)<0$, the equation $\tilde\chi(\gamma)=0$ has roots γ_1 and γ_2 on the interval $(\gamma_*-\pi,\gamma_*)$, and $\gamma_1\in(\gamma_*-\pi,\theta_0)$ and $\gamma_2\in(\theta_1,\gamma_*)$. In the case where $\tilde\chi(\gamma_*)>0$, the characteristics equation does not have real roots. Thus, satisfaction of the inequality

$$\tilde\chi(\gamma_*) < 0 \tag{3.1}$$

at the point γ_*, where $\tilde\chi'(\gamma_*)=0$ is a sufficient and necessary condition for the existence of two real roots of the characteristic equation. Below, we assume that condition (3.1) is satisfied.

We note that it is possible to formulate a simpler sufficient condition for the existence of two real roots. Indeed, if

$$g\int\limits_0^1 \frac{H'\,d\nu}{u'^2} - 1 < 0, \tag{3.2}$$

then there exist k^i ($i=1,2$) such that $\chi(k^i)=0$; in this case, $k^1\in(-\infty,\tan\,\theta_0)$ and $k^2\in(\tan\,\theta_1,\infty)$. This follows from the following properties of the function $\chi(k)$: $\chi(k)\to\infty$ as $k\to u_1$ and $k\to u_0$ and $\chi(k)<0$ for k of large absolute values by virtue of (3.2). The continuity of $\chi(k)$ implies that this function vanishes at least two points, k^1 and k^2.

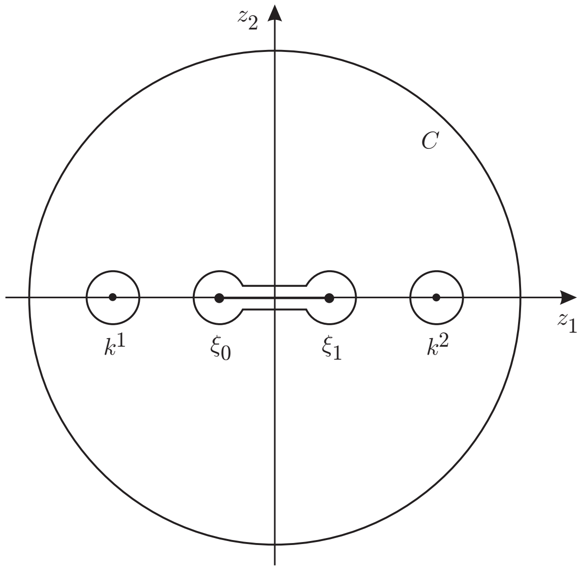

Fig. 1.

Assuming the existence of two real roots γ_1 and γ_2 on a segment of length π (and the corresponding values $k^1 = \tan\gamma_1$ and $k^2 = \tan\gamma_2$), we obtain conditions under which the characteristic equation (2.9) has no complex roots. We consider the continuation of the function χ in the plane of the complex argument z. Let $\xi_0 = \tan\theta(0)$ and $\xi_1 = \tan\theta(1)$. The function $\chi(z)$ is analytic outside the segment $[\xi_0, \xi_1]$ and has zeroes at the points $z = k^i$ $(i = 1, 2)$ and first-order poles at the points $z = \xi_0$ and $z = \xi_1$. Let us consider the contour C (see Fig. 1) including a circle of radius δ^{-1} and the contours enclosing the points $z = k^i$ and the cut $[\xi_0, \xi_1]$. The contours are located at a distance δ from the corresponding points and the cut. According to the argument principle [7], we have

$$(2\pi)^{-1}\Delta_C \arg\chi(z) = N - P,$$

where N is the number of zeroes, P is the number of poles in the region bounded by the circuit C, and $\Delta_C \arg$ denotes the increment of the argument along the contour C. Since $\chi(z)$ does not have poles outside $[\xi_0, \xi_1]$, the condition

$$\Delta_C \arg\chi(z) = 0 \tag{3.3}$$

guarantees the absence of complex zeroes in the indicated region. We note that as $\delta \to 0$, the increment of the argument tends to zero in tracing the large circle and the increments of the argument cancel each other in tracing the contours enclosing the zeroes k^1 and k^2 and the poles ξ_0 and ξ_1. At the limit, condition (3.3) becomes

$$\Delta_{[\xi_0,\xi_1]} \arg\chi^+(z)/\chi^-(z) = 0, \tag{3.4}$$

where $\chi^\pm(z)$ are the limiting values of the function $\chi(z)$ on the segment $[\xi_0, \xi_1]$ from the upper and lower half-planes. In addition, we require satisfaction of the condition

$$\chi^\pm(z) \neq 0, \qquad z \in [\xi_0, \xi_1], \tag{3.5}$$

which eliminates the neutral case where roots appear on the segment $[\xi_0, \xi_1]$.

4. Completeness of the System of Characteristic Functionals. Assuming that the hyperbolicity conditions (3.4) and (3.5) are satisfied, we study the completeness of the system of eigenfunctionals. Let the vector function $\boldsymbol{\varphi}$ satisfy the equalities

$$(\boldsymbol{F}^{1\lambda}, \boldsymbol{\varphi}) = 0, \qquad (\boldsymbol{F}^{2\lambda}, \boldsymbol{\varphi}) = 0, \qquad (\boldsymbol{F}^{3\lambda}, \boldsymbol{\varphi}) = 0, \qquad (\boldsymbol{F}^{i}, \boldsymbol{\varphi}) = 0.$$

We show that $\boldsymbol{\varphi} = 0$. From the first two equalities we obtain

$$\varphi_1 + \varphi_2 \tan\theta = 0, \qquad \varphi_{1\lambda}\tan\theta - \varphi_{2\lambda} + (v_\lambda - u_\lambda \tan\theta)\varphi_3/H = 0.$$

Taking into account that $v_\lambda - u_\lambda \tan\theta = q\theta_\lambda/\cos\theta$, $\theta_\lambda \neq 0$, we solve these relations for φ_1 and φ_3:

$$\varphi_1 = -\varphi_2 \tan\theta, \qquad \varphi_3 = H(\varphi_{2\lambda} + \varphi_2\theta_\lambda \tan\theta)/(q\theta_\lambda \cos\theta). \tag{4.1}$$

Substituting φ_1 and φ_3 into the equality

$$(\boldsymbol{F}^{3\lambda}, \boldsymbol{\varphi}) = -\frac{\tan\theta}{1+\tan^2\theta}\varphi_1 - g\tan\theta\int\limits_0^1 \frac{H'(\varphi_1'-\varphi_1)}{(v'-u'\tan\theta)^2}\,d\nu$$

$$+\frac{1}{1+\tan^2\theta}\varphi_2 + g\int\limits_0^1 \frac{H'(\varphi_2'-\varphi_2)}{(v'-u'\tan\theta)^2}\,d\nu - g\int\limits_0^1 \frac{\varphi_3'\,d\nu}{v'-u'\tan\theta} = 0$$

and performing simple transformations, we obtain the integral equation for φ_2:

$$\varphi_2 - g\int\limits_0^1 \frac{H'}{q'^2\theta'_\lambda}\frac{\partial}{\partial\nu}\frac{(1+\tan^2\theta')\varphi_2' - (1+\tan^2\theta)\varphi_2}{\tan\theta'-\tan\theta}\,d\nu = 0. \tag{4.2}$$

Let us show that this homogeneous integral equation has nontrivial solutions. We first establish that if $\boldsymbol{\varphi}_i$ is a solution of the problem

$$(B - k^i A)\boldsymbol{\varphi}_i = 0 \tag{4.3}$$

($k^i \neq \tan\theta$), then

$$(\boldsymbol{F}^{3\lambda}, A\boldsymbol{\varphi}^i) = 0. \tag{4.4}$$

Indeed, the equality

$$(\boldsymbol{F}^{3\lambda}, B\boldsymbol{\varphi}_i) = \tan\theta(\lambda)(\boldsymbol{F}^{3\lambda}, A\boldsymbol{\varphi}^i) \tag{4.5}$$

is valid because $\boldsymbol{F}^{3\lambda}$ is the eigenfunctional corresponding to the eigenvalue $\tan\theta(\lambda)$. The equality

$$(\boldsymbol{F}^{3\lambda}, B\boldsymbol{\varphi}_i) = k^i(\boldsymbol{F}^{3\lambda}, A\boldsymbol{\varphi}^i) \tag{4.6}$$

is a consequence of (4.3). Comparing (4.5) and (4.6) and taking into account that $\tan\theta(\lambda) \neq k^i$, we obtain (4.4). Simple calculations show that the eigenfunction corresponding to a root k^i of the characteristic equations (2.9) has the form

$$\boldsymbol{\varphi}^i = (k^i/(v-k^iu), -1/(v-k^iu), (1+k^{i2})H/(v-k^iu)^2).$$

From the aforesaid it is easy to see that

$$(A\boldsymbol{\varphi}^i)_2 = -u/(v-k^iu) = -1/(\tan\theta - k^i) \tag{4.7}$$

is a nontrivial solution of Eq. (4.2). The last assertion is easily verified by substituting the function (4.7) into (4.2). We consider the function

$$\phi = \varphi_2 - \sum_{i=1}^{2}\frac{\beta^i}{\tan\theta - k^i},$$

where the coefficients β^i, independent of λ, are chosen such that $\phi(0) = 0$ and $\phi(1) = 0$. Obviously, ϕ also satisfies Eq. (4.2). Integration by parts transforms the singular integral equation (4.2) to the standard form

$$(1+\xi^2)\phi\Big(\frac{1}{1+\xi^2} - \frac{gH_1}{q_1^2\theta_{1\lambda}(\xi_1-\xi)} + \frac{gH_0}{q_0^2\theta_{1\lambda}(\xi_0-\xi)} + g\int\limits_{\xi_0}^{\xi_1}\frac{\partial}{\partial\xi'}\Big(\frac{H'}{q'^2\theta'_\lambda}\Big)\frac{d\xi'}{\xi'-\xi}\Big) + g\int\limits_{\xi_0}^{\xi_1}\frac{\partial}{\partial\xi'}\Big(\frac{H'}{q'^2\theta'_\lambda}\Big)\frac{(1+\xi'^2)\phi'\,d\xi'}{\xi'-\xi} = 0.$$

We assume that the functions H and q have the derivative with respect to the variable λ that satisfies the Hölder condition, the function θ is twice differentiable with respect to λ, and its second derivative also satisfies the Hölder condition. The hyperbolicity conditions (3.4) and (3.5) guarantee that this singular equation has a unique solution in the class of Hölder functions [8]. Therefore, $\phi = 0$ and

$$\varphi_2 = \sum_{i=1}^{2}\frac{\beta^i}{\tan\theta - k^i}.$$

By virtue of (4.1), the relations $(\boldsymbol{F}^i, \boldsymbol{\varphi}) = 0$ can be written as

$$g\int\limits_0^1 \frac{H'}{q'^2\theta'_\lambda}\frac{\partial}{\partial\nu}\Big(\frac{(1+\tan^2\theta')\varphi_2'}{\tan\theta' - k^i}\Big)\,d\nu = 0.$$

Using the same reasoning as in the derivation of equality (4.4), it is easy to show that $A_{ii} \neq 0$ and

$$A_{ij} = g \int_0^1 \frac{H'}{q'^2 \theta'_\lambda} \frac{\partial}{\partial \nu} \Big(\frac{1 + \tan^2 \theta'}{(\tan \theta' - k^i)(\tan \theta' - k^j)} \Big) \, d\nu = 0 \tag{4.8}$$

for $i \neq j$. Then, from (4.1) and (4.8) it follows that $\beta^i = 0$ ($i = 1, 2$), $\varphi_2 = 0$, and $\boldsymbol{\varphi} = 0$. The completeness of the system of eigenfunctionals is proved.

5. Characteristic Form of the Equations of Motion. Let us write the equations of fluid motion in the characteristic form. For this, we sequentially act on system (1.6) with the eigenfunctionals $\boldsymbol{F}^{1\lambda}$, $\boldsymbol{F}^{2\lambda}$, $\boldsymbol{F}^{3\lambda}$, and $\boldsymbol{F}^i$. After transformations, we have

$$D\Big(\frac{q^2}{2} + gh\Big) = 0, \qquad D\Big(\frac{\theta_\lambda}{H}\Big) - \frac{2q_\lambda}{qH} D\theta = 0,$$

$$q^2 D\theta - g \tan \theta Dh + \frac{g}{\cos^2 \theta} \int_0^1 \frac{H'(q'^2 D\theta' - q^2 D\theta) \, d\nu}{q'^2 \cos^2 \theta' (\tan \theta' - \tan \theta)^2} - \frac{g}{\cos^2 \theta} \int_0^1 \frac{DH' \, d\nu}{\tan \theta' - \tan \theta} = 0,$$

$$\frac{k^i}{1 + k^{i2}} D^i h + \int_0^1 \frac{H' D^i \theta' \, d\nu}{\cos^2 \theta' (\tan \theta' - k^i)^2} - \int_0^1 \frac{DH' \, d\nu}{\tan \theta' - k^i} = 0. \tag{5.1}$$

Here $D = \partial/\partial x + \tan \theta \, \partial/\partial y$ and $D^i = \partial/\partial x + k^i \, \partial/\partial y$ are the derivatives along the characteristic directions of the continuous and discrete spectra and k^i ($i = 1, 2$) are roots of the characteristic equation (2.9). If the hyperbolicity conditions (3.4) and (3.5) are satisfied, the characteristic relations (5.1) are equivalent to Eqs. (1.6).

From Eqs. (5.1) we obtain the Bernoulli integral

$$q^2 + 2gh = q_m^2(\psi, \lambda), \tag{5.2}$$

where $q_m(\psi, \lambda)$ is an arbitrary function and $\psi(x, y, \lambda)$ (analog of the streamfunction) is defined by the relations $\psi_y = Hu$ and $\psi_x = -Hv$. In the particular case where $q_{m\psi}(\psi, \lambda)\psi_\lambda(x, y, \lambda) + q_{m\lambda}(\psi, \lambda) = 0$, from (5.2) we obtain the relation $q_\lambda(x, y, \lambda) = 0$. Then from the second equation of system (5.1) it follows that

$$D(\theta_\lambda / H) = 0, \qquad \theta_\lambda / H = A(\psi, \lambda) \tag{5.3}$$

[$A(\psi, \lambda)$ is an arbitrary function].

6. New Class of Exact Solutions. We consider the class of particular solutions of Eqs. (5.1) characterized by the equalities $q_m \equiv \text{const}$ and $A \equiv \text{const}$. From (5.3) it follows that

$$\theta = Az + \theta_0(x, y), \qquad q(x, y) = \sqrt{q_m^2 - 2gh(x, y)},$$

where $\theta_0(x, y)$ is the unknown function. Substituting the representation of the solution

$$u(x, y, z) = q(x, y) \cos (Az + \theta_0(x, y)), \qquad v(x, y, z) = q(x, y) \sin (Az + \theta_0(x, y))$$

into (1.4), we obtain the following system of equations for the unknown functions $q(x, y)$ and $\theta_0(x, y)$:

$$\tan \theta_0 q_x - q_y + q\theta_{0x} + q \tan \theta_0 \theta_{0y} = 0,$$

$$(g \tan (Ah + \theta_0) - Aq^2) q_x - (g + Aq^2 \tan (Ah + \theta_0)) q_y + gq\theta_{0x} + gq \tan (Ah + \theta_0) q_y = 0. \tag{6.1}$$

Let us determine the characteristics of system (6.1). The slopes of the characteristics are found from the quadratic equation

$$k^2 - \frac{Aq^2(\tan (Ah + \theta_0) + \tan \theta_0)}{Aq^2 - g(\tan (Ah + \theta_0) - \tan \theta_0)} k + \frac{Aq^2 \tan \theta_0 \tan (Ah + \theta_0) - g(\tan (Ah + \theta_0) - \tan \theta_0)}{Aq^2 - g(\tan (Ah + \theta_0) - \tan \theta_0)} = 0.$$

The roots of this equation

$$k_{1,2} = \frac{Aq^2(\tan (Ah + \theta_0) + \tan \theta_0)}{2(Aq^2 - g(\tan (Ah + \theta_0) - \tan \theta_0)} \pm \frac{\tan (Ah + \theta_0) - \tan \theta_0}{2(Aq^2 - g(\tan (Ah + \theta_0) - \tan \theta_0)} \sqrt{A^2 q^4 + 4Agq^2 \cot Ah - 4g^2}$$

are real provided that

$$q^2/(gh) > 2 \tan (Ah/2)/(Ah).$$

Since tan $(Ah/2) > Ah/2$ for $0 < Ah < \pi$, the given condition is stronger than the conventional condition of flow supercriticality $q^2 > gh$. The characteristic conditions

$$\theta_x + k^i\theta_x + \frac{1}{gq}\Big[-\frac{Aq^2}{2} \mp \sqrt{\frac{A^2q^4}{4} + Agq^2 \cot\, Ah - g^2}\,\Big](q_x + k^i q_y) = 0$$

are reduced to Riemann invariants. Taking into account that $h = (q_m^2 - q^2)/(2g)$, we introduce the functions

$$\mu^{1,2}(q) = \frac{1}{g}\int\limits_{q_0}^{q}\Big[\frac{Aq'}{2} \pm \sqrt{\frac{A^2q'^2}{4} + Ag\cot\,\frac{A(q_m^2 - q'^2)}{2g} - \frac{g^2}{q'^2}}\,\Big]\,dq'$$

and write the system of equations in equivalent form:

$$r_x^1 + k^1 r_x^1 = 0, \qquad r^1 = \theta - \mu^1(q),$$

$$r_x^2 + k^2 r_x^2 = 0, \qquad r^2 = \theta - \mu^2(q).$$

For $A \to 0$, this system becomes the classical shallow-water equations for stationary supercritical flows (see [9], which describes the gas-dynamic equations coinciding with the shallow-water equations for a particular polytropic exponent).

In the class of simple waves [solutions of the form of $r^1 = r^1(\alpha(x,y))$ and $r^2 = r^2(\alpha(x,y)$] Eqs. (6.1) are integrable. Following [9], for a simple wave of the first type we obtain the relations

$$r^1 = \theta_0 - \mu^1(q) = r_0^1 = \text{const}, \qquad y - k^2x = F(\theta_0),$$

and for a simple wave of the second type, the relations

$$r^2 = \theta_0 - \mu^2(q) = r_0^2 = \text{const}, \qquad y - k^1x = F(\theta_0),$$

where $F(\theta_0)$ is an arbitrary function. The above relations express that one of the Riemann invariants is constant and that the families of characteristics in the domain of definition of simple waves consists of straight lines. If the function $F(\theta_0)$ is specified, in both cases we have two equations for the two unknowns q and θ_0. Note that the existence of simple waves for the general system (1.6) was studied in [10].

In the present paper, the conditions of generalized hyperbolicity are formulated and the generalized characteristics and characteristic conditions are obtained for the system of integrodifferential equations (1.6) describing stationary long waves in free-boundary shear flow. The analysis revealed a new class of exact solutions of Eqs. (1.6), characterized by a special dependence of the unknown functions on the vertical coordinate. The simple-waves equations were integrated for the constructed special class of spatial flows.

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